Example

To illustrate the calculation of back EMF, consider the apparatus shown in **Fig. 3-6**. In this figure, the resistance of the conducting and sliding bars are lumped into the resistance R at the left. The conducting bars provide a path so that current flows through the sliding bar at any position. Passing through the loop formed by the resistance, conducting bars, and sliding bar is an applied magnetic field having a constant and uniform flux density B flowing into the page. Given this setup, it is desired to find the back EMF induced across the resistance due to sliding bar motion.

The flux flowing through the loop is given by $\phi = BLx$ where the product Lx is the area of the loop through which the magnetic field B passes. Since the loop forms a one turn coil, the flux linkage is equal to the flux itself, *i.e.*, $N = 1$, and the voltage induced is found by applying (3.14),

$$e_b = \frac{d\lambda}{dt} = \frac{d(BLx)}{dt} = BL\frac{dx}{dt} = BLv \tag{3.19}$$

where $v = dx/dt$ is the sliding bar velocity. This expression is known as the *BLv* law. The polarity of this back EMF is determined by applying Lenz's law and the right hand rule for magnetic fields about a wire.

Assume that the bar is pulled to the right, so that x is increasing. Then if the induced voltage given by (3.19) appears across the resistor with a positive poten-

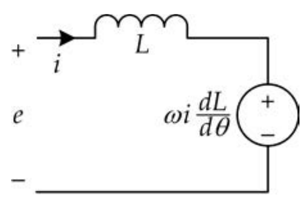

Figure 3-5. A general inductor circuit model.

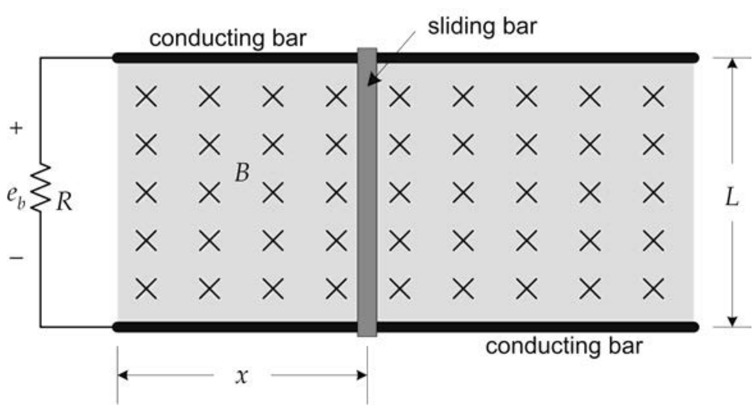

Figure 3-6. A conceptual linear motor or generator.

tial at the top, a current is induced in the loop in the counterclockwise direction. By the right hand rule, this current creates a magnetic field that is directed out of the page inside the loop. This current induced magnetic field opposes the applied magnetic field and therefore agrees with Lenz's law. Thus the voltage is positive at the top of the resistor for increasing x and an applied magnetic field directed into the page. The polarity of the induced voltage changes if either of these conditions change. If both change, *i.e.*, x decreases and the magnetic field is directed out of the page, the polarity remains the same. It is important to note that the magnetic field produced by current in the loop itself does not modify B in (3.19). The flux density B in (3.19) is external to and independent of the magnetic field produced by current flow.

Operation of this structure can also be understood from an energy perspective. Mechanical energy input from the sliding bar creates electrical energy that is converted to heat by the resistor. Although the BLv law is derived for the apparatus shown in Fig. 3-6, it is useful in many applications where a constant flux density passes through a coil. In particular, it is useful for brushless permanent magnet motor design.

3.3 Energy and Coenergy

The energy stored in a magnetic field is an important quantity to know in the design and analysis of brushless permanent magnet motors, as the magnetic field is the medium through which electric energy is converted to mechanical energy. In addition, knowing the energy or coenergy stored in a magnetic field provides one method for computing inductance.

Energy and Coenergy in Singly-Excited Systems

To illustrate the computation of energy and coenergy, reconsider the singly-excited magnetic circuit shown in Fig. 3-1. If one ignores resistive losses, the instantaneous power delivered to the magnetic field of the coil is $p=ei$ where e and i are the instantaneous voltage and current respectively in the coil forming the MMF source. Using (3.14), this can be rewritten as

$$p=i\frac{d\lambda}{dt} \tag{3.20}$$

Since power is the rate at which energy is transmitted, *i.e.*, the derivative of energy, the energy stored in the coil at a time t is given by the integral of the power

$$W=\int_0^t i\frac{d\lambda}{dt}dt=\int_{\lambda(0)}^{\lambda(t)} i\,d\lambda \tag{3.21}$$

where $\lambda(0)$ is the initial flux linkage and $\lambda(t)$ is the flux linkage at time t. For a linear magnetic circuit, i and λ are related by the inductance given in (3.4).

Substituting (3.4) into the above expression gives

$$W = \frac{1}{2L}\left[\lambda^2(t) - \lambda^2(0)\right] \tag{3.22}$$

From this expression it is apparent that if the flux linkage at time t is less than the flux linkage at time 0, the energy supplied is negative. This negative energy means that energy has come out of the magnetic field. It is customary to let the initial energy stored be zero, implying that $\lambda(0) = 0$. By doing so, the above equation describes the total energy stored in the magnetic field and (3.22) simply becomes

$$W = \frac{\lambda^2}{2L} \tag{3.23}$$

where $\lambda = \lambda(t)$.

As described by (3.21), energy stored in a magnetic field can be viewed as the shaded area to the left of the inductance line shown in **Fig. 3-7**. When $\lambda(0) = 0$ is assumed, energy is simply the area of the triangle to the left of the line.

Often times, it is convenient to express energy in terms of current rather than flux linkage as given in (3.23). For linear magnetic circuits being considered here, the area below the inductance line shown in Fig. 3-7 is equal to the area on the left. The area below the line is called coenergy and is given by

$$W_c = \int_{i(0)}^{i(t)} \lambda \, di \tag{3.24}$$

which upon substitution of (3.4) and $i(0) = 0$ becomes the familiar expression from electric circuit analysis

$$W_c = \frac{1}{2} L i^2 \tag{3.25}$$

Equations (3.23) and (3.25) define the energy and coenergy stored in a singly-excited magnetic circuit.

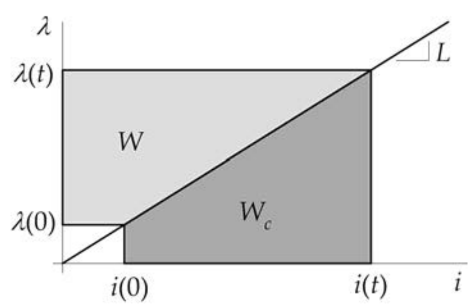

Figure 3-7. Graphical interpretation of energy and coenergy.

Before considering doubly-excited circuits, it is useful to express energy and coenergy in terms of magnetic circuit and magnetic field parameters. Since permeance is $P = \mu A/l$, flux linkage is $\lambda = N\phi$, self inductance is $L = N^2P$, and MMF is $F = Ni$, energy and coenergy can be written respectively as

$$
\begin{aligned}
W &= \frac{\lambda^2}{2L} = \frac{(N\phi)^2}{2(N^2P)} = \frac{\phi^2}{2P} \\
W_c &= \frac{1}{2}Li^2 = \frac{1}{2}(N^2P)i^2 = \frac{1}{2}PF^2
\end{aligned}
\tag{3.26}
$$

in terms of magnetic circuit parameters. In these equations, ϕ, P, and F are the flux, permeance, and MMF associated with the coil forming the inductance L.

These expressions can be related to the magnetic field parameters B, H and μ to express energy and coenergy per unit volume. Since flux is $\phi = BA$, MMF is $F = Hl$, and volume is Al, (3.26) can be manipulated to give the energy and coenergy densities

$$
\begin{aligned}
w &= \frac{W}{Al} = \frac{\phi^2}{2PAl} = \frac{(BA)^2}{2(\mu A/l)Al} = \frac{B^2}{2\mu} \\
w_c &= \frac{W_c}{Al} = \frac{PF^2}{2Al} = \frac{1}{2Al}(\mu A/l)(Hl)^2 = \frac{\mu H^2}{2}
\end{aligned}
\tag{3.27}
$$

Energy and Coenergy in Doubly-Excited Systems

For doubly-excited systems such as that shown in Fig. 3-2, expressions for energy and coenergy are more involved because energy is stored in both the self and mutual inductances. In particular, the calculation of energy stored in mutual inductance requires more rigor than the preceding analysis. As a result, only the final result is given here and the interested reader is encouraged to consult other references.

The instantaneous power delivered to the magnetic field in Fig. 3-2 is

$$
p = i_1\frac{d\lambda_1}{dt} + i_2\frac{d\lambda_2}{dt}
\tag{3.28}
$$

where the subscripts refer to the respective coils. From this expression, the energy stored in the magnetic field is

$$
W = \frac{\lambda_{11}^2}{2L_1} + \frac{\lambda_{22}^2}{2L_2} + \frac{\lambda_{12}\lambda_{21}}{L_{12}}
\tag{3.29}
$$

where $\lambda_{11} = N_1\phi_{11}$, $\lambda_{22} = N_2\phi_{22}$, $\lambda_{12} = N_1\phi_{12}$, and $\lambda_{21} = N_2\phi_{21}$. The coenergy follows as